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# *The Method of Graphs applied to Compound Partitions.*

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If we divide a number $N$ into two portions in all possible ways, and then partition each of the portions in all possible ways, we shall form all the possible bi-partitions of $N$. If the number of portions is three we form all the tri-partitions of $N$; and if the number of portions is unlimited we shall form all the compound partitions of $N$. Let

$$a_1a_2\ldots a_\alpha \,|\, b_1b_2\ldots b_\beta \,|\, c_1c_2\ldots c_\gamma \,|\, \ldots \,|\, e_1e_2\ldots e_\epsilon$$

be such a compound partition of $N$. I shall call this a regular compound partition of $N$ if the following conditions are satisfied:

$$a_i \nless b_i \nless c_i \ldots \nless e_i,$$

whatever $i$ may be.

Now all such regular compound partitions may be represented graphically by an array of points in space as follows: let each of the portions be represented by an array of points in a plane and then let the planes be superimposed: the plane containing the first portion on top and that containing the second portion next, etc.; first lines lying above first lines and first columns above first columns. Then it is evident, that in general, any compound partition may be read in six different ways: that is, to any given compound partition there are five others which are conjugate. For example, the compound partitions conjugate to

531|211|11 are 32211|31|2; 521|311|11;
32111|311|2; 332|21|2|1|1 and 332|21|11|1|1.

The following method may be given for obtaining the five compound partitions which are the conjugates of any given compound partition: the first one of the conjugates is obtained by writing for each portion of the given compound partition its conjugate single partition—thus 32211 is the conjugate of 531: the second of the conjugates is obtained from the given compound partition by taking the first of the elements of each portion of the given compound partition to form the first portion of the new conjugate, and the second of the elements of each portion to form the second portion, etc.: the third and fifth conjugates are obtained from the second and fourth conjugates respectively, in the same manner as the first conjugate was obtained from the given compound partition: finally, the fourth conjugate is obtained from the first conjugate in the same manner that the second conjugate was obtained from the given compound partition; or it may be obtained from the given compound partition directly by an extension of Professor Sylvester's method of calling off—thus 1 goes in 3 elements of the first portion of the given compound partition, in 3 elements of the second portion and in 2 elements of the third portion, giving 332 for the first portion of the fourth conjugate; again, 2 goes in 2 elements of the first portion and in 1 element of the second portion, giving 21 for the second portion of the fourth conjugate, etc.

The points lie in three sets of parallel planes. If the array of points in each of the planes of one of the three sets be symmetrical, then there will be three conjugates instead of six, as for example, the three following compound partitions are conjugates 531|311|11; 32211|311|2; 332|21|21|1|1.

In addition there are the self-conjugate compound partitions in which the array of points in each of the planes of all three sets of parallel planes is symmetrical. Among the smaller numbers these are not very numerous: 1 has the self-conjugate partition 1; 4 has the self-conjugate compound partition 21|1; and there are no more examples till we come to 7, which has the two self-conjugate compound partitions 311|1|1 and 22|21. And it is easy to see that in the cases where there is no cube equal to or less than the number to which the number is congruous with respect to the modulus 3, there can be no self-conjugate compound partition.* Thus the following numbers are void of self-conjugate compound partitions: 2, 3, 5, 6, 9, 12, 15, 18, 21, 24; and all other numbers have self-conjugate compound partitions.

If we use the symbol $(w;\ n;\ i,\ j,)$ to signify the number of ways in which we can divide the number $w$ into regular compound partitions: the number of portions being not more than $n$, each portion being partitioned into $i$ or fewer parts not greater than $j$, then we evidently have

$$(w;\ n;\ i,\ j,) = (w;\ n;\ j,\ i,) =$$
$$(w;\ i;\ n,\ j,) = (w;\ i;\ j,\ n,) =$$
$$(w;\ j;\ n,\ i,) = (w;\ j;\ i,\ n,)$$

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* This is, it may be remarked, *en passant*, exactly similar to the case of single partitions in which there is no self-conjugate partition for a number $n$, if there be no square equal to or less than $n$, to which the number is congruous with respect to the modulus 2: and the number 2 is the only number fulfilling that condition.

Of this we may notice a special example, namely, if $n = 2$ and $i = j = w$, we have all the regular bi-partitions of $w$. Then we shall have $(w;\ 2;\ w,\ w,) = (w;\ w;\ 2,\ w,) = (w;\ w;\ w,\ 2,)$ that is, the number of regular bi-partitions of any number, $w$, is equal to the number of compound partitions of $w$ in which the number of parts in which any portion is partitioned is not greater than two, and also to the number of compound partitions of $w$ in which no part in any portion is greater than two.

It may also be noticed that we may have like results in the case of what I shall call regular $n$-compound partitions. For example, all the bi-compound partitions of any number $N$ are obtained by writing all the single partitions of $N$ and then partitioning each of the elements of each single partition of $N$ into compound partitions. Then if

$$a_{11}a_{12}\ldots a_{1\alpha_1}|b_{11}\ldots b_{1\beta_1}|\ldots|e_{11}\ldots e_{1\epsilon_1}||a_{21}a_{22}\ldots a_{2\alpha_2}|b_{21}\ldots b_{2\beta_2}|\ldots$$
$$|e_{21}\ldots e_{2\epsilon_2}||\ldots||a_{j_1}a_{j_2}\ldots a_{j\alpha_j}|b_{j_1}\ldots b_{j\beta_j}|\ldots|e_{j_1}\ldots e_{j\epsilon_j}$$

be a bi-compound partition of $N$, it is termed regular if the following conditions are satisfied

$$a_{ik} \nless b_{ik} \nless c_{ik} \nless \ldots \nless e_{ik}$$

and $$d_{1n} \nless d_{2n} \nless d_{3n} \nless \ldots \nless d_{jn},$$ where

$i$, $k$, $n$ have any values and $d$ means any of the letters $a$, $b$, $c \ldots e$. Then we can represent any such bi-compound partition by an array of points in four-fold space and accordingly have in general twenty-four conjugates, which may, in special cases, reduce to twelve or four. There will also be self-conjugate bi-compound partitions in the cases in which the number $N$ is such that there is some number equal to or less than $N$ which is the fourth power of some integer, to which fourth power $N$ is congruous with respect to the modulus 4. For all numbers then of the form $4n + 2$ or $4n + 3$ and for the numbers 4, 8 and 12, there are no self-conjugate bi-compound partitions. If the symbol

$$(w;\ m;\ n;\ i,\ j,)$$

be used to signify the number of ways in which we can divide the number $w$ into regular bi-compound partitions, the number of apportionments being not more than $m$, the number of portions in each apportionment not greater than $n$, each portion being partitioned into $i$ or fewer parts not greater than $j$, then we have of course $(w;\ m;\ n;\ i,\ j,) = (w;\ m_1;\ n_1;\ i_1,\ j_1,)$,
where the numbers $m_1$, $n_1$, $i_1$, $j_1$, are the numbers $m$, $n$, $i$, $j$, in any of the 24 possible orders of arrangement.